Nenita Sornito

Teaching Mathematics the Integrated Way

Learning is said to be more lasting when acquired through many avenues. Miss Sornito demonstrates how mathematical concepts can be made intelligible through different approaches.

[After teaching] high school mathematics for four years and college mathematics for eight years, the writer has become aware of the state of unpreparedness of high school graduates for the study of college mathematics. She is, therefore, of the conviction that there is an urgent need for remedial measures to improve our mathematics instruction in the high scrool. Unless this is done, the teaching of the subject can not accomplish its objectives, namely, (1) to provide young people with a powerful tool in industry and in their daily living, and (2) to provide a discipline that will develop their mental capacities to the optimum degree.

There are a number of areas open for investigation in order to accomplish these objectives, such as (1) *The Teacher-centered Area* (upgrading the teacher's preparation), (2) *The Student-centered Area* (a study of the student's fitness for, and difficulties in, the study of mathematics in relation to their latent mental capacities), (3) *The Curriculum-centered Area* (re-examination of our mathematics curriculum and offerings in order to update them, and (4) *The Classroom-centered Area* (developing and improving the methods and techniques of teaching the subject in order to improve the effectiveness of instruction). The researcher limits her investigation to the fourth area, by experimenting on the use of a specific pattern or technique herein called the teaching of integrated or unified mathematics. This idea was conceived by the writer during her years of teaching and observing, and its basic idea is shared by many mathematics teachers and textbook writers abroad. This is evidenced by recent college textbooks like *Integrated Algebra and Trigonometry* by Vance,[1] *Elements of Calculus and Analytic Geometry* by Thomas,[2] and many others.

While some efforts at integration in college mathematics are now in progress, the writer is not aware of similar attempts made at the high school level here in the Philippines, although many mathematics teachers have expressed opinions in favor of the idea, hence this experiment.

(Please turn to page 16)

[1]Elbridge P. Vance, ***Integrated Algebra and Trigonometry***. (Reading, Mass.: Addison - Wesley Publishing Company, Inc., 1960.)

[2]George B. Thomas, Jr., ***Elements of Calculus and Analytic Geometry***. (Reading, Mass.: Addison - Wesley Publishing Company, Inc., 1959.) .

Nenita Sornito...
(Continued from page 15)

Procedure

ORGANIZATION OF THE CLASSES. Two experiments were conducted, the first during the first semester of 1962-1963 and the second during the first semester of 1963-1964. In each experiment two classes each consisting of 35 students were constituted; one was identified as the control group and the other, the experimental group. For convenience, the experiment was conducted in a Mathematics 110 (Review of High School Mathematics and Solid Geometry) class, with students in the first year of the College of Engineering at Central Philippine University as cases. These classes were made as identical as was possible under the circumstances. This was done by administering a survey test to determine the general background of the students.

THE VARIABLE FACTORS. Other variable factors than the teaching techniques involved were next considered. Among these were (1) the age factor, (2) the sex factor, (3) the study load factor, and (4) the socio-economic factor. A careful examination of the situation revealed that the influence of these other variables was nullified; that is, they were practically held constant in so far as the groups were concerned. The time of classes and the environmental conditions were also equalized. After these were considered, the experimenter was satisfied that the variables were practically controlled and there was nothing left to influence the results except the teaching pattern which was the subject of the investigation.

TEACHING OF THE CLASS. The classes were taught by the researcher herself. The decision to do so was dictated by the desire to eliminate the problem of looking for two teachers who are truly comparable. The experimenter, however, was very well aware of the danger of a bias in the teaching when the teacher favors one of the factors to be tested. At the very outset she made it a point to adopt the scientific attitude. She did her best to equalize everything about her teaching in both classes except for the patterns used, the traditional pattern in the control group, and the integrated pattern in the experimental group. She did not have much difficulty with the teaching of the traditional pattern, since she had been following this for a number of years. In the teaching of the integrated pattern, however, the researcher had to collect her teaching materials from recent books and current mathematics journals and had to seek the assistance of her adviser in the organization of her teaching materials. These materials were used to supplement the basic text.

The researcher realized the fact that it is just impossible to effect complete integration. According to Professor Fehr, in order to do this, it would be necessary to "establish a new set of postulates, for surely those of Euclidian geometry are distinct and based on different elements than those of algebra." At present, there is no such set of postulates that the writer is aware of. The researcher, therefore, had to resort to other means and ways of teaching the experimental group, namely, (1) employing algebra and geometry simultaneously; that is, using both the algebraic and geometric approach in teaching fundamental concepts like the theory of sets, functional concepts, and others, and (2) using algebra in the teaching of geometry and vice-versa.

EXAMPLES OF INTEGRATION. The following are some of the examples of integration:

1. *The teaching of signed numbers.* To beginners in elementary algebra, signed numbers mean nothing more than numbers preceded by plus (+) or minus (-) signs. This is practically what most high school graduates would give as definition. Such a definition would be like calling the picture of a man, or the written name of a man, the man himself. Some teachers would correct this misconception by telling the class that numbers preceded by the plus sign (positive) are numbers greater than zero (0) and numbers preceded by the minus sign (negative) are less than zero (0). Some may say that they are directed numbers. Both of these so-called corrections or further explanation, though correct, would still be very vague, and would leave the students still wondering.

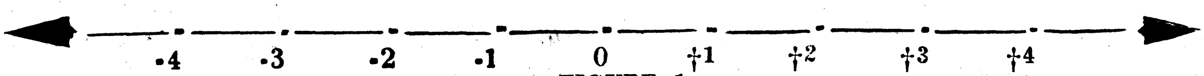

FIGURE 1
DIRECTED LINES

But if the definition or explanation is clarified by the use of directed lines (Figure 1), then the students will begin to attach some significance to positive (+) and negative (-) numbers by looking at the picture. This would be just the same as showing the picture of a man instead of trying to make a verbal or written description of his face.

2. *The process of multiplication.* Multiplication in arithmetic can be translated into proportion in al-
(Please turn to page 31)

Nenita Sornito...
(Continued from page 16)

gebra, and properties of similar triangles in geometry.

Given:

Line a = ______________

Line b = ______________

Required: To multiply two lines, that is, to find a line which which is the product of the line a and b.

Procedure: Draw two lines M and N meeting at O (Figure 2), and measure a unit length OX. Measure OY = a and XZ = b. Draw XY and through Z draw a line parallel to XY meeting ON at W. YW is equal to ab, the product of line a and b.

Proof:

1:a = b:YW A line parallel to one side of a triangle divides the other sides into segments that are proportional.

YW = ab In any proportion, the product of the means is equal to the product of the extremes.

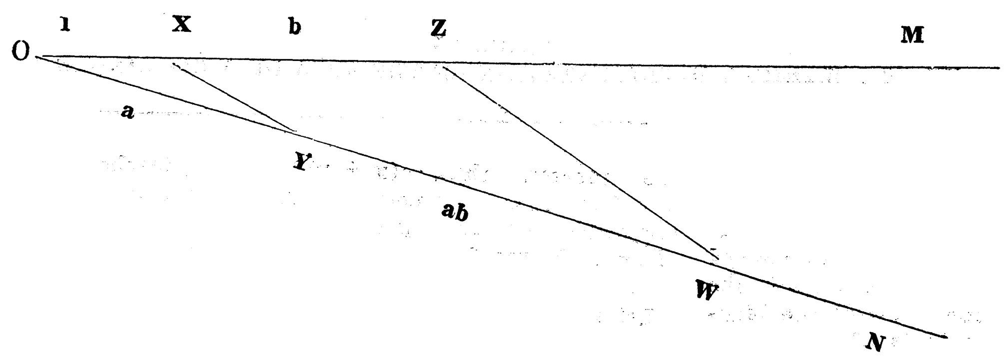

FIGURE 2

GEOMETRICAL REPRESENTATION OF MULTIPLICATION

3. *Finding the area of a rectangle.* The elementary concept for any rectangle is that, area is the product of the measure number of the length and the measure number of the width. This is applied in arithmetic on rectangles of different dimensions, thus: for a rectangle of length 4 units and width 3 units (Figure 3), the area is 12 square units; for a rectangle of length 6 units and width 4 units, the area is 24 square units; and for a rectangle of length 5 units, width 2 units, the area is 10 square units, and so forth.

In each of the above examples, the dimensions are specific and so are the corresponding areas. (This is arithmetic).

The above specific cases can be generalized and expressed in a single pattern, by designating a symbol to represent the length of any rectangle and another symbol to represent the width of the same rectangle. Thus, in Figures 4a, 4b, and 4c, if L and W are designated to represent the length and width respectively of any rectangle, then area equals length times width (A = LW). (This is algebra or

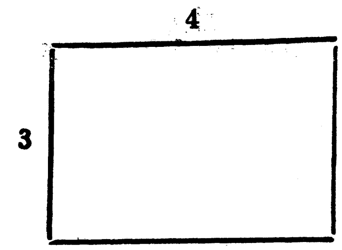

FIGURE 3

ARITHMETICAL REPRESENTATION OF THE AREA OF A RECTANGLE

"generalized" arithmetic).

In geometry, one defines area as the number of unit squares that the rectangle contains (Figures 5a and 5b).

From the above-mentioned figures, it is shown that there are 3 (or W) rows of 4 (or L) squares each, thus making the area 4 x 3 squares = 12 squares or L x W squares = LW squares.

4. *Illustrating the square of a binomial.* The general pattern of teaching the square of a binomial in algebra is to give a

Please turn to page 32)

Nenita Sornito...
(Continued from page 31)

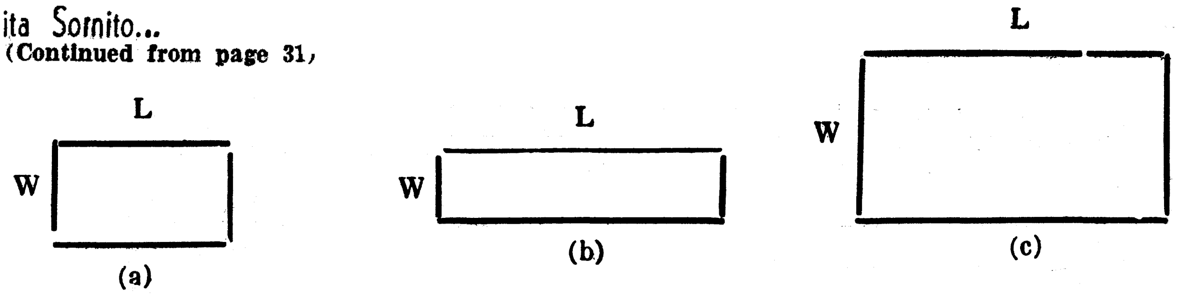

FIGURE 4

ALGEBRAIC REPRESENTATION OF THE AREA OF A RECTANGLE

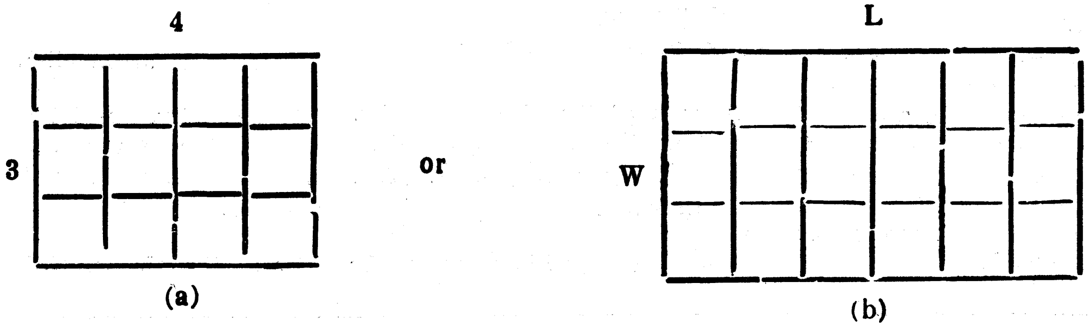

FIGURE 5
GEOMETRICAL REPRESENTATION OF THE AREA OF A RECTANGLE

few examples such as $(a + b)^2$, $(x + y)^2$, and so forth. Then after the regular process of long multiplication the answers are studied for the relation of the terms in the product to the terms of the binomial. On the basis of this observation a rule is established in the form of an empirical pattern for

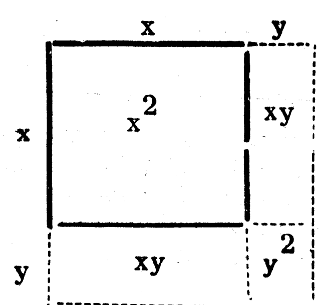

FIGURE 6
GEOMETRICAL REPRESENTATION OF THE SQUARE OF A BINOMIAL

the process, thus, $(x + y)^2 = x^2 + 2xy + y^2$. A geometrical demonstration of this principle is given in Figure 6.

Let x be the side of the original square. Then the area is x^2. If the sides are increased by y to make them $(x+y)$, then the total area is $x^2+xy+xy + y^2$ or $x^2 + 2xy + y^2$.

5. *The process of evolution.* Any high school graduate should have no difficulty in extracting the square root of any positive number. But when students are asked to explain why they multiply the root already found by 20 to get the trial divisor, their only answer is, "Our high school teacher in mathematics gave that to us." In the traditional pattern of teaching this operation, students are arbitrarily told no more than that they should multiply the root already found by 20 to get the next trial divisor. The teacher concentrates on the proper manipulation of the process.

The only way to explain this this step in the process is by referring back to the binomial expansion in algebra; that is, $(a + b)^2 = a^2 + 2ab + b^2$, in which "a" stands for the 10's digit and "b" stands for the units digit. After obtaining the first digit of the root, $2ab + b^2$ remains. In order to obtain the next digit this remainder is first factored into $b(2a + b)$. Since "a" stands for the 10's digit, the trial divisor should be (2 x 10a) instead of (20 x a) as is the common practice. Here is where algebra explains an arithmetical process.

The writer has not yet met a high school graduate who can extact the cube root of a number. But referring again to the "bino-

(Please turn to page 33)

Nenita Sornito...

(Continued from page 32)

mial theorem," this can be reduced to the pattern: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. After the first digit of the root is taken, the remainder is $3a^2b + 3ab^2 + b^3$.

Factoring this remainder into $b(3a^2 + 3ab + b^2)$, the quantity in parenthesis is obtained as the complete divisor. The value $3a^2$ is used for obtaining the trial divisor, but since "a" stands for the 10's digit, therefore the trial divisor becomes $3(10a)^2$. By this method, the students begin to learn the process of evolution in both arithmetic and algebra.

Geometry can also be used in extracting the square root of a line.

Given: Line a = ____________

Required: To extract the square root of the given line a.

Procedure: On the line OC (Figure 7), measure OA equal to one unit length. Measure AB equal to the given line a. Draw AD perpendicular to OC. With OB as diameter, draw a semi-circle intersecting AD at E. AE is the square root of the given line a.

Proof:

$1 : AE = AE : a$. In a right triangle the altitude upon the hypotenuse is the mean proportional between the segment of the hypotenuse.

$AE = \sqrt{a}$ In any proportion, the product of the means is equal to the product of the extremes.

6. *The process of involution.* Another demonstration on how an arithmetic and/or algebraic process can be interpreted geometrically is the fundamental operation of involution (raising a power).

Involution is simply continued multiplication by the same number, that is, $2^5 = 2 \times 2 \times 2 \times 2 \times 2$, and $(a+b)^4 = (a+b)(a+b)(a+b),(a+b)$, and can be performed easily in both arithmetic and algebra. Sornito[3] has demonstrated this process geometrically using two different methods, both based upon the simple properties of right triangles, as shown in the following (Figures 8 and 9):

Given:

Line a = ____________

Required: To find the nth power of a given line a.

Procedure: (First method)

(Please turn to page 34)

---

[3]Juan E. Sornito, "Involution Operated Geometrically." *The Mathematics Teacher,* 48:243-244, April, 1955.

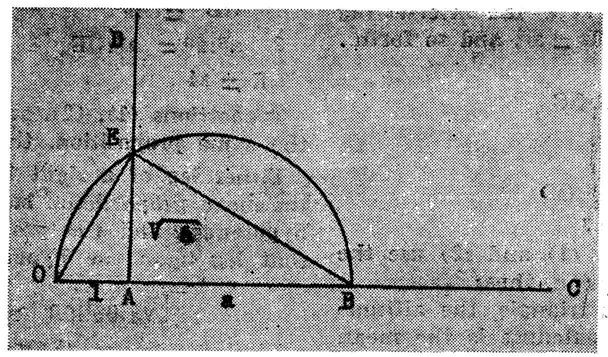

FIGURE 7
GEOMETRICAL REPRESENTATION OF EXTRACTING OF SQUARE ROOT

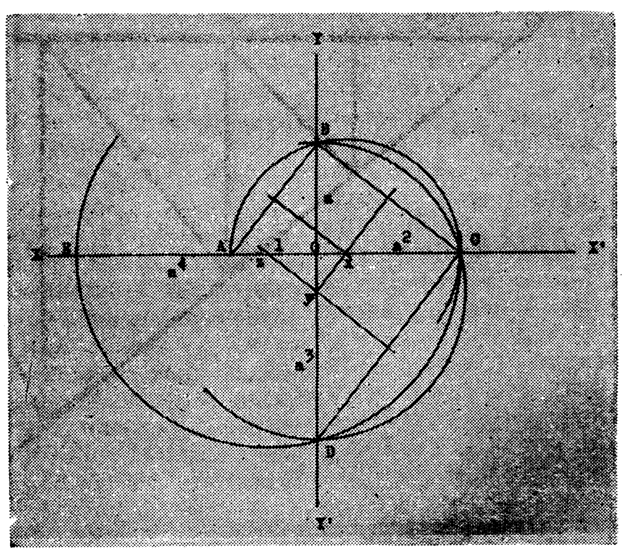

FIGURE 8
GEOMETRICAL REPRESENTATION OF INVOLUTION (FIRST METHOD)

Nenita Sornito...

(Continued from page 33)

Draw two lines perpendicular to each other as shown in Figure 8. Measure AO = 1 unit length.

Measure OB $= a$. Construct a perpendicular bisector to AB, meeting XX' at x. With x as center, describe a circle passing through A and B and intersecting XX' at C. OC $= a^2$. Construct again a perpendicular bisector to the line BC, intersecting YY' at y. With y as center, describe a circle passing through B and C and intersecting YY' at D. OD $= a^3$, and so forth..

Proof:

1. $1:a = a:OC$

   $OC = a^2$

2. $a:a^2 = a^2:OD$

   $OD = a^3$

Proportions (1) and (2) use the same proposition, thus:

In a right triangle, the altitude upon the hypotenuse is the mean proportional between the segments of the hypotenuse.

Procedure: (Second Method)

Let OA on the line OM equal the unit length. At A erect a perpendicular to OM. By the use of a compass measure OB equal to a, the given line. Through B draw ON. Construct BC perpendicular to ON; CD perpendicular to OM; DE perpendicular to ON; EF perpendicular to OM; and so forth. It can be proved that OC $= a^2$, OD $= a^3$, OE $= a^4$, and so forth, to a^n.

Proof:

1. $1:a = a:OC$

   $OC = a^2$

2. $a:a^2 = a^2:OD$

   $OD = a^3$

3. $a^2:a^3 = a^3:OE$

   $OE = a^4$

Proportions (1), (2), and (3) use the same proposition, thus:

Either leg of a right triangle is the mean proportional between the hypotenuse and the projection of that leg upon the hypotenuse.

Evaluation

Final testing at the end of the term. Upon the termination of the term, a final test was administered to the two groups for the purpose of determining their relative achievements. The same sets of questions were used consisting of two types; namely, (1) exercises to test their accuracy and skill in the mechanical operations, and (2) problems to test their power to interpret, that is, their ability to translate ideas into the language of mathematics, and their ability to make complete solutions.

The papers were carefully graded, and from the results the achievements of the students in the two groups were compared. These results were carefully tabulated and statistically analyzed.

Evaluating the results of the final test. Since one is continually trying to answer questions, especially in scientific work, one may wish to answer such a specific question as this: "Is the integrated teaching of algebra and geometry superior to the traditional pattern presently followed in the secondary schools?"

Findings

The evaluation of the comparative merits of the two types of teaching approaches was based

(Please turn to page 38)

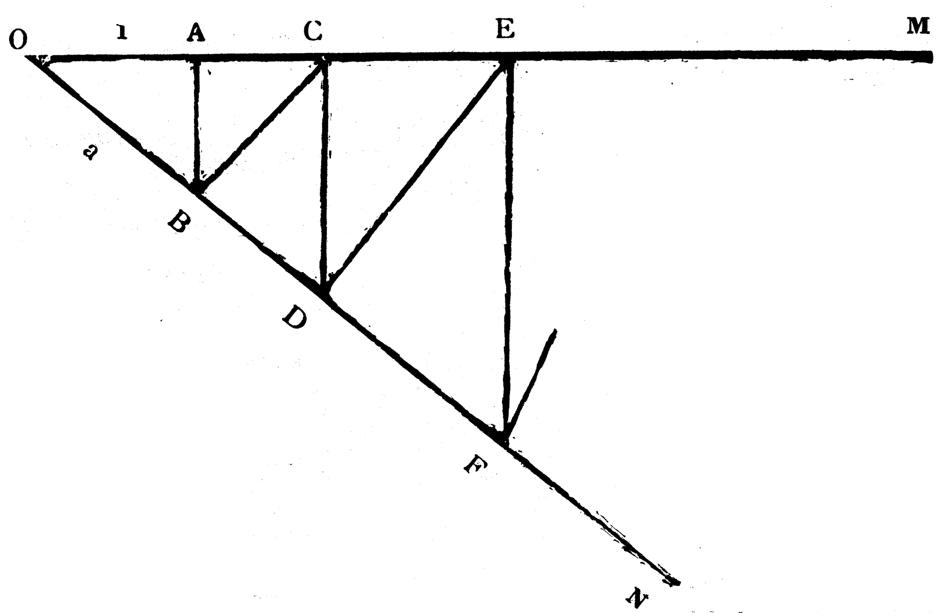

FIGURE 9
GEOMETRICAL REPRESENTATION OF INVOLUTION (SECOND METHOD)

Nenita Sornito...

(Continued from page 34)

primarily upon:

(1) The observations made during the course of each experiment. Those were on the responses of the students, their readiness to understand new concepts, and their ability to apply them to problems.

(2) The statistical analysis of the results of the final test.

In the former, the new head of the department of mathematics who was also the faculty adviser of the researcher was invited from time to time to observe in the classes and make the evaluation. In the latter, the null hypothesis was employed to determine the comparative progress of the control group and the experimental group respectively. From this analysis the results given below were obtained.

Since the corresponding values of t are greater than 1.9968 which is the value required for the 5% level of confidence at 68 degrees of freedom, the null hypothesis is rejected; the two different variables in question, the teaching methods, produced significant differences, and the integrated pattern of teaching is proved to be more effective than the traditional pattern of teaching mathematics.

Test Group	No. of Cases	Mean	Standard Error of the Mean	Standard Deviation
		1962-1963 FINAL TEST		
Experimental	35	93.06	25.32	4.34
Control	35	80.57	23.63	4.05
Value of the t-ratio = 2.11, significant for 68 D.F. at the 5% level of confidence				
		1963-1964 FINAL TEST		
Experimental	35	90.11	24.08	4.13
Control	35	77.34	23.05	3.95
Value of the t-ratio = 2.23, significant for 68. D.F. at the 5% level of confidence				

Conclusions

While the experimenter recognized the limitations of the experiment in point of time and scope, she submits the following conclusions:

1. That the integrated teaching of mathematics (geometry and algebra) can give the students a broader perspective of the science of mathematics.

2. That this pattern of instruction can increase the students' ability to correlate the different areas of the science thereby increasing their power to use it as a tool in industry and in their daily living.

3. That because of the first two observations, integrated teaching can be more motivating and challenging.

4. That integrated teaching opens more opportunity for repetition of basic principles, thus reducing to a minimum the element of forgetfulness.

5. Finally, that the adoption of this pattern and technique of the mathematics instruction in teaching mathematics will upgrade the secondary schools.

Recommendations

On the basis of the observations and findings made in this experiment, the researcher submits the following recommendations:

1. That the integrated pattern of teaching the basic concepts of algebra and geometry be adopted in the first year of high school.

2. That a continuing experimental program be undertaken by institutions of higher learning, particularly the teacher training colleges, for the purpose of studying further the merits of this pattern of instruction.

3. That a study committee be created to prepare outlines, syllabi, and other materials for the type and pattern of teaching in this experiment.

4. Lastly, it is strongly recommended that Central Philippine University, where this experiment was conducted, initiate a movement to implement the foregoing recommendations.